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Mh4714 Week 11

Week 11

0.1 Integration (contd.)

If $f(x) < 0 \quad \forall x \in [a, b]$ then it is easy to see that $\int_a^b f$ is the negative of the area between the curve, the x-axis and the lines $x = a$ and $x = b$.
If $f(x)$ is sometimes positive and sometimes negative over $[a, b]$ it is easy to see that $\int_a^b f$ is the area under the curve above the x-axis minus the area above the curve, and below the x-axis, and the lines $x = a$ and $x = b$.

Note: The notation $\displaystyle\int_a^b f(x)\mathrm{d}x$ has evolved from the special form that upper and lower sums have when $f(x)$ is continuous and the partition uses n sub-intervals of equal length.
If the length of each sub-interval is Δx then we can write

$$U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = \sum_{i=1}^{n} M_i \Delta x,$$
$$L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \sum_{i=1}^{n} m_i \Delta x.$$

Since f is continuous then we will have $M_i = f(x_i^*)$ for some $x_i^* \in [x_{i-1}, x_i]$ and $m_i = f(x_i^{**})$ for some $x_i^{**} \in [x_{i-1}, x_i]$. That is

$$U(f, P) = \sum_{i=1}^{n} M_i \Delta x = \sum_{i=1}^{n} f(x_i^*) \Delta x, L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \sum_{i=1}^{n} f(x_i^{**}) \Delta x.$$

and then

$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^{**}) \Delta x.$$

The symbol $\int$ is the ghost of the $\sum$ symbol and the dx symbol is the ghost of the Δx.
The symbol dx is useful because it keeps track of the variable of integration. For example

$$\int_1^2 x^2 y \mathrm{d}x \text{ and } \int_1^2 x^2 y \mathrm{d}y$$

are different integrals.

0.1.0.1 Properties of Integrals. If $f(x)$ and $g(x)$ are both integrable over $[a, b]$ then

(i) $\displaystyle\int_a^b f(x) + g(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_a^b g(x)\mathrm{d}x.$

(ii) $\displaystyle\int_a^b kf(x)\mathrm{d}x = k\int_a^b f(x)\mathrm{d}x$ for any $k \in \mathbb{R}$.

Theorem 0.1 (Fundamental Theorem of Calculus)

Let f be integrable over $[a, b]$ and $f = g'$ for some function g, then

$$\int_a^b f(x)dx = g(b) - g(a)$$

Proof

Let $P = \{x_0, \ldots, x_n\}$ be any partition of $[a, b]$

Applying the Mean Value Theorem to the function g over the interval $[x_{i-1}, x_i]$, we know that there is a point $c_i \in [x_{i-1}, x_i]$ with

$$g(x_i) - g(x_{i-1}) = g'(c_i)(x_i - x_{i-1}) = f(c_i)(x_i - x_{i-1})$$

Using the usual notation we have

$$m_i \leq f(c_i) \leq M_i$$

$$\Rightarrow m_i(x_i - x_{i-1}) \leq f(c_i)(x_i - x_{i-1}) \leq M_i(x_i - x_{i-1})$$

$$\Rightarrow m_i(x_i - x_{i-1}) \leq g(x_i) - g(x_{i-1}) \leq M_i(x_i - x_{i-1})$$

$$\Rightarrow L(f,P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \leq \sum_{i=1}^{n}(g(x_i) - g(x_{i-1})) \leq \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = U(f,P)$$

But

$$\sum_{i=1}^{n}(g(x_i) - g(x_{i-1})) = (g(x_1) - g(x_0)) + (g(x_2) - g(x_1)) + (g(x_3) - g(x_2)) \dots (g(x_n) - g(x_{n-1}))$$

$$= g(x_n) - g(x_0) = g(b) - g(a).$$

That is

$$L(f,P) \leq g(b) - g(a) \leq U(f,P) \text{ for every partition } P$$

That is $g(b) - g(a)$ is an upper bound for $\mathfrak{L}$ and a lower bound for $\mathfrak{U}$. That is

$$\text{lub } \mathfrak{L} \leq g(b) - g(a) \leq \text{glb } \mathfrak{U}$$

and since f is integrable over $[a,b]$ we have lub $\mathfrak{L} =$ glb $\mathfrak{U} = \int_a^b f(x)dx$
That is $g(b) - g(a) = \int_a^b f(x)dx$.

□

Example 0.2

Since $\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{3}x^3 = x^2$ it follows from the above theorem that

$$\int_a^b x^2 \mathrm{d}x = \frac{1}{3}b^3 - \frac{1}{3}a^3.$$

0.1.0.2 Indefinite integral.

Because of the above theorem, the symbol $\int f(x)\mathrm{d}x$ without the upper and lower limits represents an *anti-derivative* of $f(x)$..
That is,

$$\int f(x)\mathrm{d}x = F(x) \Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}F(x) = f(x).$$

Since $\frac{\mathrm{d}}{\mathrm{d}x}k = 0$ for any $k \in \mathbb{R}$ there can be infinitely many anti-derivatives for a given function.

Example 0.3

$$\int x^3 \mathrm{d}x = \tfrac{1}{4}x^4 + c \quad \text{for any } c \in R.$$

It is also true that

$$\int \left(\frac{\mathrm{d}}{\mathrm{d}x} f(x) \right) \mathrm{d}x = f(x) + c, \quad \text{for any } c \in \mathbb{R}.$$

0.1.1 Techniques of Integration

0.1.1.1 Integration by Substitution.

The *substitution rule* is based on the *chain rule* for differentiation.

Recall that, according to the chain rule (assuming that all functions are suitably differentiable), we have

$$\frac{\mathrm{d}}{\mathrm{d}x} f(u(x)) = \frac{\mathrm{d}}{\mathrm{d}u} f(u) \frac{\mathrm{d}}{\mathrm{d}x} u(x)$$

Example 0.4

$$\frac{\mathrm{d}}{\mathrm{d}x} \sin(x^2) = 2x \cos(x^2)$$

Therefore we can see that

$$\int \left(\frac{d}{du} f(u) \frac{d}{dx} u(x) \right) \mathrm{d}x = \int \frac{\mathrm{d}}{\mathrm{d}x} (f(u(x))) \mathrm{d}x = f(u(x)).$$

Example 0.5

$$\int 2x \cos(x^2) \mathrm{d} = \int \frac{\mathrm{d}}{\mathrm{d}x} \sin(x^2) \mathrm{d} = \sin(x^2).$$

The *substitution rule* is now obvious because

$$\int \left(\frac{d}{du} f(u) \frac{d}{dx} u(x) \right) \mathrm{d}x = f(u(x)) = \int \frac{d}{du} f(u) \mathrm{d}u.$$

Notationally we see that

$$\frac{d}{dx} u(x) \mathrm{d}x$$

in the left hand integral has been replaced by

$$\mathrm{d}u$$

in the right hand integral (as if dx has been cancelled!)
Thus we get the *substitution rule*

$$\int F(u)\frac{\mathrm{d}}{\mathrm{d}x}u(x)\mathrm{d}x \to \int F(u)\mathrm{d}u.$$

Example 0.6

(i)

$$\int \overset{\frac{du}{dx}}{\overset{\downarrow}{2x}} \cos(\overset{u}{\overset{\downarrow}{x^2}})\mathrm{d}x = \int \cos(u)\mathrm{d}u = \sin(u) + c = \sin(x^2) + c$$

Therefore

$$\int_a^b 2x\cos(x^2)\mathrm{d}x = \sin(b^2) - \sin(a^2).$$

(ii)

$$\int (b+x)^n\mathrm{d}x = \int \overset{\frac{du}{dx}}{\overset{\downarrow}{1}}\,(\overset{u}{\overset{\downarrow}{b+x}})^n\mathrm{d}x = \int u^n\mathrm{d}u = \frac{u^{n+1}}{n+1} = \frac{(b+x)^{n+1}}{n+1} + c$$

if $n \neq -1$.
Therefore

$$\int_a^b (b+x)^n\mathrm{d}x = \left[\frac{(b+x)^{n+1}}{n+1}\right]_a^b = \frac{2b^{n+1}}{n+1} - \frac{(b+a)^{n+1}}{n+1}$$

if $n \neq -1$.

(iii)

$$\int (b-x)^n\mathrm{d}x = -\int \overset{\frac{du}{dx}}{\overset{\downarrow}{-1}}(\overset{u}{\overset{\downarrow}{b-x}})^n\mathrm{d}x = -\int u^n\mathrm{d}u = -\frac{u^{n+1}}{n+1} = -\frac{(b-x)^{n+1}}{n+1} + c$$

if $n \neq -1$.
Therefore

$$\int_a^b (b-x)^n\mathrm{d}x = \left[-\frac{(b-x)^{n+1}}{n+1}\right]_a^b = \frac{(b-a)^{n+1}}{n+1}$$

if $n \neq -1$.

0.1.1.2 Integration by Parts.
The product rule for differentiation suggests another technique for evaluating indefinite integrals:

Firstly note the fact that $\displaystyle\int \frac{\mathrm{d}}{\mathrm{d}x} f(x)\mathrm{d}x = f(x) + \text{ constant }$ by definition of indefinite integration.

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x}+v\frac{\mathrm{d}u}{\mathrm{d}x} \Rightarrow \int \frac{\mathrm{d}}{\mathrm{d}x}(uv)\mathrm{d}x = \int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}+\int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x \Rightarrow uv = \int \frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x+\int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$$

$$\Rightarrow = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x.$$

We thus transform the integral $\displaystyle\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x$ into $\displaystyle\int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$ and it sometimes happens that $\displaystyle\int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$ is easier to evaluate than $\displaystyle\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x$.

Example 0.7

- Evaluate $\displaystyle\int x\cos(x)\mathrm{d}x$.

  We note that we can easily find an indefinite integral for $\cos(x)$ and we can reduce x to 1 by differentiation.

  $$\int \underset{\substack{\shortparallel\\ u}}{x}\underset{\substack{\shortparallel\\ \frac{dv}{dx}}}{\cos(x)}\mathrm{d}x = \underset{\substack{\shortparallel\\ u}}{x}\underset{\substack{\shortparallel\\ v}}{\sin(x)} - \int \underset{\substack{\shortparallel\\ \frac{du}{dx}}}{1}\ \underset{\substack{\shortparallel\\ v}}{\sin(x)}\mathrm{d}x = x\sin(x) - \int \sin(x)\mathrm{d}x$$

  $$= x\sin(x) - (-\cos(x)) + \text{ constant } = x\sin(x) + \cos(x) + \text{ constant}$$

- Evaluate $\displaystyle\int x^2\cos(x)\mathrm{d}x$.

  Here we will apply integration by parts twice:

  $$\int \underset{\substack{\shortparallel\\ u}}{x^2}\underset{\substack{\shortparallel\\ \frac{dv}{dx}}}{\cos(x)}\mathrm{d}x = \underset{\substack{\shortparallel\\ u}}{x^2}\underset{\substack{\shortparallel\\ v}}{\sin(x)} - \int \underset{\substack{\shortparallel\\ \frac{du}{dx}}}{2x}\underset{\substack{\shortparallel\\ v}}{\sin(x)}\mathrm{d}x$$

  We now apply integration by parts again to evaluate $\displaystyle\int 2x\sin(x)\mathrm{d}x$.

  $$\int \underset{\substack{\shortparallel\\ u}}{x}\underset{\substack{\shortparallel\\ \frac{dv}{dx}}}{\sin(x)}\mathrm{d}x = \underset{\substack{\shortparallel\\ u}}{x}\underset{\substack{\shortparallel\\ v}}{(-\cos(x))} - \int \underset{\substack{\shortparallel\\ \frac{du}{dx}}}{1}\ \underset{\substack{\shortparallel\\ v}}{(-\cos(x))}\mathrm{d}x$$

$$= -x\cos(x) + \int \cos(x)\mathrm{d}x = -x\cos(x) + \sin(x) + \text{ constant}$$

Therefore we have:

$$\int x^2\cos(x)\mathrm{d}x = x^2\sin(x) - 2(-x\cos(x) + \sin(x)) + \text{ constant}$$
$$= x^2\sin(x) + 2x\cos(x) - 2\sin(x)) + \text{ constant}$$

0.1.2 Taylor's Theorem

We can prove a very important theorem using integration by parts.

Theorem 0.8

Let f be infinitely differentiable over $\mathbb{R}$.
Prove that for any real numbers a and b

$$f(b) = f(a) + f^{(1)}(a)(b-a) + \frac{f^{(2)}(a)}{2!}(b-a)^2 + \frac{f^{(3)}(a)}{3!}(b-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \int_a^b \frac{f^{(n+1)}(x)}{n!}(b-x)^n\mathrm{d}x$$

for all integers $n \geq 0$.

Proof

We will use induction to prove this.
To prove the case $n = 0$, note that

$$\int_a^b \frac{f^{(1)}(x)}{0!}(b-x)^0\mathrm{d}x = \int_a^b f^{(1)}(x)\mathrm{d}x = f(b) - f(a)$$

That is,

$$f(b) = f(a) + \int_a^b \frac{f^{(1)}(x)}{0!}(b-x)^0\mathrm{d}x.$$

which is the case for $n = 0$.

Now we will show that the $n = k$ case implies the $n = k+1$ case:

$$f(b) = f(a) + f^{(1)}(a)(b-a) + \frac{f^{(2)}(a)}{2!}(b-a)^2 + \frac{f^{(3)}(a)}{3!}(b-a)^3 + \cdots + \frac{f^{(k)}(a)}{k!}(b-a)^k + \int_a^b \frac{f^{(k+1)}(x)}{k!}(b-x)^k\mathrm{d}x$$

Using integration by parts we get

$$\begin{aligned}
\int_a^b \frac{f^{(k+1)}(x)}{k!}(b-x)^k \mathrm{d}x \\
&= \left[\frac{f^{(k+1)}(x)}{k!}\left(-\frac{(b-x)^{k+1}}{k+1}\right)\right]_a^b - \int_a^b \frac{f^{(k+2)}(x)}{k!}\left(-\frac{(b-x)^{k+1}}{k+1}\right)\mathrm{d}x \\
&= \frac{f^{(k+1)}(a)}{(k+1)!}(b-a)^{k+1} + \int_a^b \frac{f^{(k+2)}(x)}{(k+1)!}(b-x)^{k+1}\mathrm{d}x
\end{aligned}$$

And so we have

$$\begin{aligned}
f(b) = f(a) + f^{(1)}(a)(b-a) + \frac{f^{(2)}(a)}{2!}(b-a)^2 + \frac{f^{(3)}(a)}{3!}(b-a)^3 + \cdots + \frac{f^{(k)}(a)}{n!}(b-a)^k \\
+ \frac{f^{(k+1)}(a)}{(k+1)!}(b-a)^{k+1} + \int_a^b \frac{f^{(k+2)}(x)}{(k+1)!}(b-x)^{k+1}\mathrm{d}x
\end{aligned}$$

Therefore it follows from the Principle of Induction that

$$\begin{aligned}
f(b) = f(a) + f^{(1)}(a)(b-a) + \frac{f^{(2)}(a)}{2!}(b-a)^2 + \frac{f^{(3)}(a)}{3!}(b-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n \\
+ \int_a^b \frac{f^{(n+1)}(x)}{n!}(b-x)^n \mathrm{d}x
\end{aligned}$$

for all $n \geq 0$. $\square$